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SOME LOCI AND THEIR PROJECTIONS.

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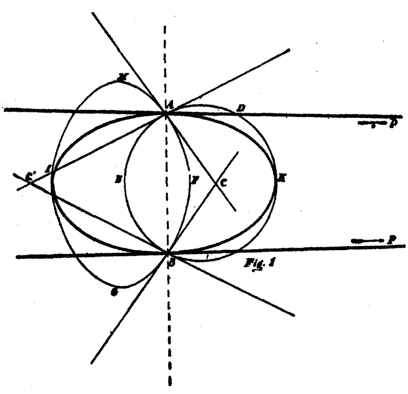
If a conic be described through two fixed points, A and B, and touching two given conics which also pass through those points, the locus of the of pole AB with respect to the varying conic is a conic touching the four lines CA, CB, $C'A$, and $C'B$, where C and C' are the poles of AB with respect to the two given conics.

At first reading this proposition or problem seems very complicated and difficult of solution; and, indeed, it would be if the ordinary methods of analytical geometry were used. But treated from the standpoint of projective or the so-called modern geometry, the solution is very easy and leads to some interesting results. This proposition and the one deduced from it later in this article are but examples of a class which lends itself very easily to projective methods.



In figure 1, let A and B be the two given points. Let the two fixed conics, $ADKB$ (called S) and $AGIH$ (called S') intersect at the given points. There will then be a doubly infinite number of conics which may pass through the two given points and be tangent to the conics S and S'. In figure 1, the heavy lined conic $AKBI$ is one of these conics. (There will be a set which will be tangent at about the points E and F). Draw the tangents AP

and BP. The point of intersection, P, will be the pole of the variable conic with respect to the line AB. As the conic $AKBI$ takes each of its infinite number of positions, the point P will trace a curve which will touch the tangent lines CA, $C'A$, CB, and $C'B$.

Let us project A and B into the focoids or circular points at infinity (*Scott's Modern Analytical Geometry*, Art 201). The line AB will be projected into the line at infinity (*Salmon's Conic Sections*, Art. 254). Now since a circle is the only conic which cuts the line at infinity at the focoids, the conics S, S' and $AKBI$ must be projected into circles when the points A and B are projected intc the focoids (*Scott's Modern Analytical Geometry*, Arts. 117-118). The poles of the line AB will now be at the centers of these circles (*Smith's Conic Sections*, Art. 314). Hence, after projection, the above proposition would read: "The locus of the centers of circles tangent to two given circles is a conic tangent to CA, CB, $C'A$, and $C'B$."

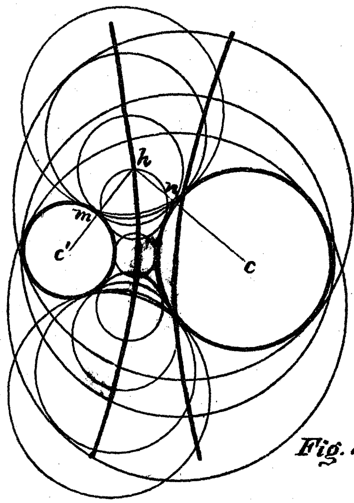


Fig. 2

In figure 2, let the heavy circles be the two given ones (or, the ones into which S and S' are projected). There will be four sets of tangent circles. Figure 2 shows those which lie outside of both the given circles and those which include both circles in their areas. The locus of the centers of these two sets will be an hyperbola, one set making one branch and the second set making the other. The centers of the given circles are the foci of the locus. That this curve is an hyperbola may be easily proved. Let h (Fig. 2) be any point on the curve; let r and r' be the radii of the given circles. Then since $hn = hm$, $hc - hc' = r - r' =$ a constant.

From the definition of the curve the locus will be an hyperbola. Moreover this curve will be tangent to the lines connecting c and c' with the focoids or circular points at infinity (*Salmon's Conic Sections*, Art. 258; *Scott's Modern Analytical Geometry*, Art. 129).

Since this property is a descriptive one (*Scott's Modern Analytical Geometry*, Chapter V), it will be true after projection and the only difference between the proposition given at the beginning of this article and the one just proved is in the location of the points A and B. In the latter the circular points at infinity are the given points, while in the former any two points may be taken. The former is a general proposition, the latter a special case. In this special case we say "circle" instead of "conic through two fixed points." We say "center" for "pole of the line AB with respect to the conic," and again "hyperbola" for "conic touching the four lines CA, CB, $C'A$, and $C'B$.

Referring again to figure 1, we see that there may be two cases, one in which the two given conics intersect in real points and the other in which they

have an imaginery intersection. The two cases after projection are illustrated in figures 2, 3, 4 and 5. In figure 3, one conic lay within the other so that after projection one circle will be within the other and our locus is easily proved to be an ellipse with C and C' the centers of the given circles as foci. Moreover the ellipse is tangent to the lines connecting C and C' with the focoids (*Scott's Modern Analytical Geometry*, Arts. 129-130; *Salmon's Conic Sections*, Art. 258) Figures 4 and 5 show the case in which the circles after projection, intersect in real points. One set of circles will give an hyperbola and the other an ellipse as the locus. If in figures 2 and 3, the two given circles have equal radii, the locus of the light lined circles will be a straight line perpendicular to the line connecting the centers.

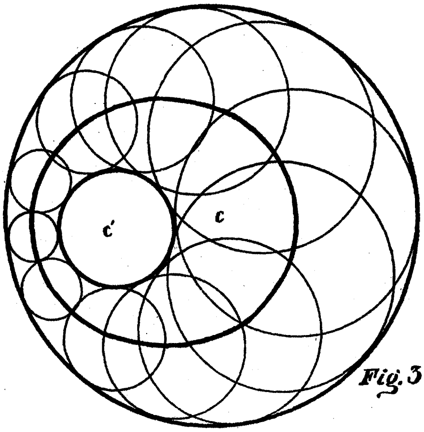


Fig. 3

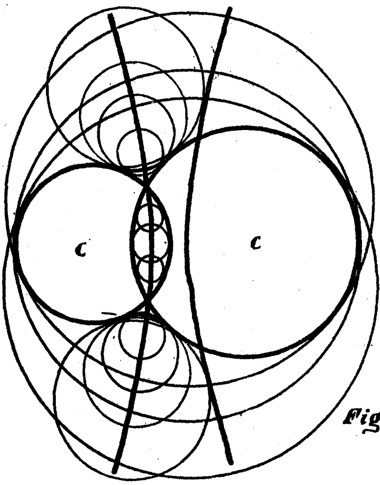


Fig. 4

If now we imagine one of the conics in figure 1 contracted to a point we shall have another proposition, viz:

If a conic be described tangent to a given conic and passing through two fixed points on that conic, and also passing through some other given point, the locus of the pole with respect to the varying conic of the chord connecting the first two points is a conic tangent to the lines connecting them with the third point, and also tangent to the lines connecting them with the pole of the chord with respect to the given conic.

This is another seemingly complicated and difficult proposition, but it is easily proved by projection. In figure 6, let A and B be the two points on the conic $ACDB$. Let P be the third given point. An infinite number of conics may be drawn through A, B and P and tangent to $ACDB$. The heavy lined conic in figure 6 is one of these. We must prove then that the locus of M as the variable conic changes is a conic tangent to PA, PB, NA and NB. As before, project A and B into the focoids and $ACDB$ in-

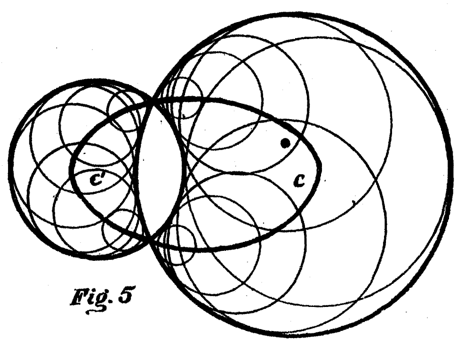


Fig. 5

to a circle. The system of conics through P, A, and B and tangent to $ACDB$ will then be a system of circles.

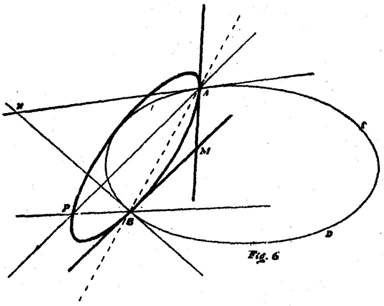


Fig. 6

There will be two cases in this proposition. We may take the given point within or without the conic. Figures 7 and 8 show the two cases after projection. P and C are the projections of P and N respectively. The proposition will now read: "The locus of the centers of a system of circles passing through a given point and tangent to a given circle is a conic having the given point and the center of the given circle for foci." In figure 7 when P is within the circle the locus is an ellipse. In figure 8, the locus is an hyperbola. We may have a third case if we imagine P upon the circumference of the circle. In that case the locus is a straight line through C and P.

Having proved this descriptive property after projection, it must be true before. Hence our general proposition is proved, and in the same manner that figures 2, 3, 4, and 5 are special cases of figure 1, so are figures 7 and 8 special cases of figure 6.

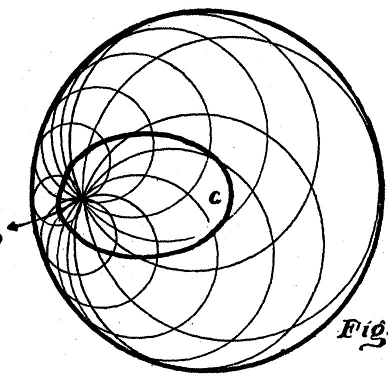


Fig. 7

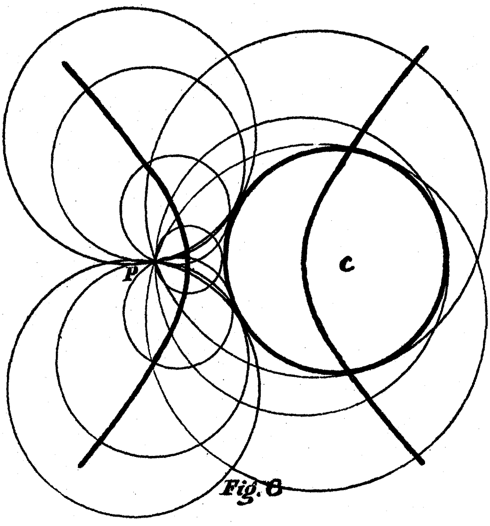


Fig. 8